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# THE DYNAMICS OF A RIGID BODY UNDER IMPACT*

V.A. SINITSYN

The motion of an absolutely rigid body under impact (impulsive motion) is considered. The analogy between this motion and the motion of a rigid body in a fluid is pointed out: the influence of the inertial properties of the body on the motion is defined in both cases by three second-order surfaces. The role of these surfaces when a body moves in infinite fluid was established by Zhukovskii /1/. Using the moments of the impact pulses at the point of contact (the influence of rolling friction and revolving friction), the necessary conditions are obtained for the appearance of "tangential" impact (TI). A well-known characteristic of TI is that the reaction assists in increasing the approach velocity of the points of contact of the colliding bodies, including the case when the initial approach velocity is zero ("collision without impact"). Previously /2-4/ studies of TI have only taken account of the impact pulse (the normal component of the reaction and sliding friction). The physical meaning of TI has been elucidated in a discussion of the "paradoxes" of dry friction, see /5, Appendix 2/, and in the popular literature /6/. In working devices TI often makes its appearance as dynamic selfbraking, and as unwanted cases of "sticking" and "seizing."

We consider the motion of a rigid body which belongs to a system with ideal constraining links, linear with respect to the velocities (holonomic or non-holonomic). Let the impact action on the body be specified as a principal vector $\mathbf{S}$ and principal momentum $\mathbf{L}$ of the impact force momenta, reduced to some centre. As usual in the case of impact, we neglect displacements of the material particles of the system. As the basic coordinate system we take a fixed system whose origin coincides with the centre of reduction of the impact force momenta. Determination of the motion of the rigid body under impact amounts to finding the angular velocity $\boldsymbol{\omega}$ of the body and the velocity $\mathbf{v}$ of some pole.

As the pole we take the point $O$ of the body which coincides with the origin of the basic coordinate system (the position of the pole remains fixed during the impact, while its velocity varies from some value $\mathbf{v}^-$ before impact). We write the equations of the impulsive motion of the body (motion under impact) /7/

$$v_i - v_i^- = \frac{\partial \Phi_1}{\partial S_i} + \frac{\partial \Phi_3}{\partial S_i}, \quad \omega_i - \omega_i^- = \frac{\partial \Phi_2}{\partial L_i} + \frac{\partial \Phi_3}{\partial L_i} \tag{1}$$

$$2\Phi_1 = S^T\alpha S,\ 2\Phi_2 = L^T\beta L,\ \Phi_3 = S^T\gamma L$$

$$\alpha = \|\alpha_{ij}\|,\ \beta = \|\beta_{ij}\|,\ \gamma = \|\gamma_{ij}\|\ (i, j = 1, 2, 3)$$

Here, $v_i, \omega_i, S_i, L_i$ $(i = 1, 2, 3)$ are the projections of the vectors $\mathbf{v}, \boldsymbol{\omega}, \mathbf{S}, \mathbf{L}$ on the axes of the fixed coordinate system. The coefficients of the matrices $\alpha, \beta, \gamma$ are found by means of expressions for the kinetic energy $\Theta$ of the reduced system (a quadratic form in the kinetic

*Prikl.Matem.Mekhan.*, 51, 1, 9-14, 1987 energy, expressed in terms of generalized velocities) (a is the matrix of inertial coefficients of the reduced system, and $n$ is the number of independent generalized velocities $q_1^{\cdot}, \ldots, q_n^{\cdot}$)

$$(q^{\cdot} - q^{\cdot -})^T a (q^{\cdot} - q^{\cdot -}) = 2\Theta(q_1^{\cdot} - q_1^{\cdot -}, \ldots, q_n^{\cdot} - q_n^{\cdot -}) = 2\Phi_1 + 2\Phi_2 + 2\Phi_3 \tag{2}$$

With the functions $\Phi_1, \Phi_2, \Phi_3$ we associate three second-order surfaces

$$\chi_\delta: \ \delta_{11}x^2 + \delta_{22}y^2 + \delta_{33}z^2 + 2\delta_{23}yz + 2\delta_{31}zx + 2\delta_{12}xy = \chi_\delta(x, y, z) = 1 \quad (\delta = \alpha, \beta, \gamma) \tag{3}$$

The surfaces $\chi_\alpha, \chi_\beta$ are ellipsoids, since, with $\mathbf{L} = 0$ and $\mathbf{S} = 0$, the functions $\Phi_1$ and $\Phi_2$ are respectively equal to twice the kinetic energy of the acquired velocities of the reduced system (2).

We shall show that the role of the surfaces $\chi_\delta$ in the description of the impulsive motion of the body is similar to the role of the three surfaces in the motion of a rigid body in a fluid /1/ (we shall apply the names used by Zhukovskii to similar concepts).

If the increment of the velocity of point $O$ is directed along the radius $\mathbf{r}_1$ of the ellipsoid $\chi_\alpha$, then the momentum $S$ and velocity increment $\Delta v$ in this direction are connected by the relation

$$\Delta v = \frac{S}{2r_1^2}\left(\frac{\partial\chi_\alpha}{\partial x}x + \frac{\partial\chi_\alpha}{\partial y}y + \frac{\partial\chi_\alpha}{\partial z}z\right) = \frac{S}{r_1^2} \qquad (r_1^2\Delta v = S) \tag{4}$$

By analogy, $\mu = r_1^2$ will be called the changed mass of the body in the direction $\mathbf{r}_1$. It follows from (4) that the ellipsoid $\chi_\alpha$ is obtained by marking off a distance $r_1 = \sqrt{\mu}$ from the point $O$ in the direction $\mathbf{r}_1$. The changed mass in any direction is not less than the body mass, since, by Kelvin's theorem /8/, the kinetic energy of the acquired velocities of a system with links (in the case of preassigned increments ahead of the indicated points) is not less than the kinetic energy of the acquired velocities of a rigid body without links.

If the impact action on the body is represented solely by an impulsive couple, whose plane is perpendicular to the radius $r_2$ of the ellipsoid $\chi_\beta$ and communicates in the direction $\mathbf{r}_2$ an angular velocity increment $\Delta\omega$, then

$$\Delta\omega = \frac{L}{2r_2^2}\left(\frac{\partial\chi_\beta}{\partial x}x + \frac{\partial\chi_\beta}{\partial y}y + \frac{\partial\chi_\beta}{\partial z}z\right) = \frac{L}{r_2^2} \quad (r_2^2\Delta\omega = L) \tag{5}$$

By analogy, $\nu = r_2^2$ will be called the changed moment of inertia with respect to the $\mathbf{r}_2$ axis. The ellipsoid $\chi_\beta$ is obtained by marking off a distance $r_2 = \sqrt{\nu}$ in the direction $\mathbf{r}_2$ from the point $O$ (the ellipsoid of changed moments of inertia). On changing the pole, the ellipsoid of changed moments of inertia varies, but the changed moment of inertia with respect to any axis is never less than the moment of inertia of the rigid body about this axis (this is again proved by Kelvin's theorem).

By choosing the pole position we can arrange for the coefficients $\gamma_{ij}$ to be symmetrical with respect to the subscripts /1/. By analogy, we call this point central. For the central point, the surface $\chi_\gamma$ is an ellipsoid or hyperboloid. We draw the radius vector $\mathbf{r}_3$ to a point of this surface. If the moment of the impulsive couple $\mathbf{L}$ is directed along $\mathbf{r}_3$, then, in addition to the angular velocity increment, which can be found from (5), an increment of the velocity (of the central point) $(\mathbf{S} = 0)$ occurs in the direction $\mathbf{r}_3$

$$\Delta v = \frac{L}{2r_3^2}\left(\frac{\partial\chi_\gamma}{\partial x}x + \frac{\partial\chi_\gamma}{\partial y}y + \frac{\partial\chi_\gamma}{\partial z}z\right) = \frac{L}{r_3^2} \quad (\lambda\Delta v = L, \lambda = r_3^2)$$

An conversely, if the momentum S acts along $\mathbf{r}_3$, in addition to the velocity increment there is an angular velocity increment

$$\Delta\omega = S/r_3^2, \ \lambda\Delta\omega = S \ (\lambda = r_3^2)$$

If the surface $\chi_\gamma$ is a hyperboloid, then, in the direction of the generators of the asymptotic cone, the increments of the velocity of the central point and of the angular velocity of the body are zero.

In short, as in the case of the influence of a rigid body in a fluid on the motion communicated to it /1/, the entire influence of a rigid body and a system with ideal constraining links on the impulsive motion is characterized by the three surfaces (3). These surfaces, and the surfaces considered in /1/, are mutual.

Let us consider the conditions under which tangential impact (TI) /3/ occurs when a rigid body moves along a surface with friction. We assume that the body and surface are convex (there is a unique common tangent plane and a common normal at the point of contact), and that there are no active impact forces. We assume that the surface reaction consists of a force /2, 3/ and a couple. We locate the origin of the basic coordinate system at the point of contact, with the first two unit vectors of the basis in the tangent plane, and the third along the common normal within the body. We introduce as independent variable $\tau$ in the equations of motion the momentum of the normal component $N$ of the principal reaction vector ($d\tau = Ndt$). Then, for the component $v_3$ (where $-v_3$ is the approach velocity) we obtain from (1) the equation

$$dv_3/d\tau = \alpha_{31}X + \alpha_{32}Y + \alpha_{33} + \gamma_{31}M_x + \gamma_{32}M_y + \gamma_{33}M_z \quad (6)$$

where $X, Y, M_x, M_y, M_z$ are the projections of the force of friction, the moment of rolling friction and the moment of revolving friction, referred to the absolute value of the reaction normal component.

To find the coefficients on the right-hand side of (6), we write the expression for the kinetic energy (2)

$$\Theta = \frac{S^2}{2m} + \frac{1}{2\triangle^2}(A_1M_1^2 + B_1M_2^2 + C_1M_3^2 + 2D_1M_2M_3 + 2E_1M_1M_3 + 2F_1M_1M_2) \quad (7)$$

$$\triangle = ABC - AD^2 - BE^2 - CF^2 - 2DEF, \quad S^2 = S_1^2 + S_2^2 + S_3^2$$

$$A_1 = Aa^2 + Bf^2 + Ce^2 - 2Def - 2Eae - 2Faf$$

$$D_1 = Aef + Bbd + Ccd - D(bc + d^2) - E(cf + de) - F(be + df)$$

$$(A_1B_1C_1,\ D_1E_1F_1;\ ABC,\ DEF,\ abc,\ def)$$

$$a = BC - D^2,\ d = AD + EF\ (abc,\ def;\ ABC,\ DEF)$$

$$\mathbf{M} = \mathbf{L} - \mathbf{r} \times \mathbf{S}; \quad \mathbf{M} = \| M_1, M_2, M_3 \|^T, \quad \mathbf{r} = \| \xi, \eta, \zeta \|^T$$

Here, $\mathbf{M}$ and $\mathbf{r}$ are the principal moment of the reaction momenta with respect to the centre of mass and the radius vector of the position of the centre of mass in the basic coordinate system, $A, B, C, D, E, F$ are the components of the inertia tensor for the centre of mass in axes parallel to the axes of the basic coordinate system and $m$ is the mass of the body. The relations not written are obtained by circular permutation of the symbols shown in parentheses.

From (7) we obtain the coefficients

$$\alpha_{31} = \triangle^{-2}(-B_1\xi\zeta + D_1\xi\eta - E_1\eta^2 + F_1\eta\zeta) \quad (8)$$

$$\alpha_{32} = \triangle^{-2}(-A_1\eta\zeta - D_1\xi^2 + E_1\xi\eta + F_1\xi\zeta)$$

$$\alpha_{33} = m^{-1} + \triangle^{-2}(A_1\eta^2 + B_1\xi^2 - 2F_1\eta\zeta)$$

$$\gamma_{31} = \triangle^{-2}(-A_1\eta + F_1\xi), \quad \gamma_{32} = \triangle^{-2}(B_1\xi - F_1\eta), \quad \gamma_{33} = \triangle^{-2}(D_1\xi - E_1\eta)$$

We introduce the mapping point with coordinates $x = X$, $y = Y$ on the tangent plane to the body surfaces. Then, the necessary conditions for the approach velocity to increase (negative right-hand side of Eq.(6)) is that the straight line and circle must intersect /3/ ($f_*$ is the ratio of the force of friction to the normal component of the reaction)

$$\alpha_{31}x + \alpha_{32}y + \alpha_{33} + \gamma_{31}M_x + \gamma_{32}M_y + \gamma_{33}M_z = 0, \quad x^2 + y^2 = f_*^2 \quad (9)$$

The circle is the section of the circular cone whose axis is along the normal to the surfaces, $f_* \leqslant f_0$ ($f_0$ is the coefficient of sliding friction; the equation is obtained only in the case of pure sliding, when there is no revolution and the cone is the cone of friction).

There is as yet no adequate model of the interaction of rigid bodies which takes account of the dependence of the rolling and turning resistance moments on the normal reaction momentum and the state of the body during collision. The usual approach, based on using the coefficients of rolling friction and revolving friction, obtained empirically for pure rolling and pure turning, does not in general /9/ give a realistic description of the motion.

By using differential Eq.(6), the conditions for intersection of the stright line and circle (9),

$$f_* > |(\alpha_{33} + \gamma_{31}M_x + \gamma_{32}M_y + \gamma_{33}M_z)|\,(\alpha_{31}^2 + \alpha_{32}^2)^{-1/2} \quad (10)$$

and the expressions for the coefficients (8), we can prove some properties of the approach and the conditions under which TI occurs (impact resulting from an increase of the approach velocity even when the initial approach velocity is zero).

1°. Rolling friction and revolving friction do not affect the variation of the approach velocity, and TI is impossible for any finite values of the coefficient of sliding friction ($f_0 > 0$), if the centre of mass of the body is on the normal to the surface ($\xi = \eta = 0$), since in this case $dv_3/d\tau = \alpha_{33} > 0$.

2°. Rolling friction does not affect the variation of the approach velocity (or the conditions for TI to occur), if we have ($\gamma_{31} = \gamma_{32} = 0$)

$$\xi/\eta = A_1/F_1 = F_1/B_1$$

3°. Revolving friction does not affect the variation of the approach velocity if ($\gamma_{33} = 0$)

$$D_1\xi - E_1\eta = 0$$

A direct check shows that $\gamma_{33} = 0$ in particular when one of the principal diameters of the central ellipsoid of inertia is perpendicular to the tangent plane to a surface at the point of contact.

4°. For the special orientation of the rigid body indicated in Para.3° with the extra condition $\zeta = 0$ (the centre of mass is located on the tangent plane), the variation of approach velocity is determined solely by the rolling friction, and TI can arise when we have (the necessary condition)

$$\frac{1}{m} + \frac{\eta}{A}(\eta - M_x) + \frac{\xi}{B}(\xi + M_y) < 0$$

If the rolling friction is found as usual /10/ with the aid of the coefficient of rolling friction $k$ (the moment of the couple is taken to be opposite to the angular velocity of rolling), i.e.,

$$M_x = -k\omega_1/|\omega|, \quad M_y = -k\omega_2/|\omega|, \quad |\omega| = (\omega_1^2 + \omega_2^2)^{1/2}$$

then TI cannot arise due to rolling friction when $k$ is sufficiently small, in accordance with the inequality

$$ABm^{-1} + (A\xi^2 + B\eta^2) > k(A^2\xi^2 + B^2\eta^2)^{1/2} \tag{11}$$

If the inequality (11) is of opposite sign, there is a sector of directions of angular velocity of rolling in which

$$k\left(-\frac{\eta\omega_1}{A} + \frac{\xi\omega_2}{B}\right)\frac{1}{|\omega|} > \frac{1}{m} + \frac{\eta^2}{A} + \frac{\xi^2}{B}$$

Consequently, regardless of the principal vector of friction forces and the revolving friction couple $(\alpha_{31} = \alpha_{32} = 0,\ \gamma_{33} = 0)$, TI is caused solely by rolling friction.

5°. The effect of turning on the conditions for TI to occur shows itself also in a variation of the coefficient $f_*$. If we take account of a small finite area of contact surface, it can be shown /9/ that, for motions close to pure revolution, the coefficient $f_*$ depends linearly on the ratio $(v_1^2 + v_2^2)^{1/2}/|\omega_3 r|$ ($r$ is the radius of the locally spherical contact surface). Hence, for sufficiently large angular velocity of revolution, we can in general avoid the occurrence of TI (a change in the sign of the inequality in (10)).

6°. In the case of plane motion of a rigid body parallel to the coordinate plane which passes through the common normal, we have the following expressions for the coefficients (8):

$$\alpha_{31} = 0, \quad \alpha_{32} = -\frac{\eta\xi}{m\rho^2}, \quad \alpha_{33} = \frac{1}{m} + \frac{\eta^2}{m\rho^2}$$

$$\gamma_{31} = -\frac{\eta}{m\rho^2}, \quad \gamma_{32} = \gamma_{33} = 0 \quad (m\rho^2 = A)$$

In this case, condition (10) becomes

$$f_* > |\zeta^{-1}(\rho^2\eta^{-1} + \eta - M_x)| \tag{12}$$

If we neglect the moment of rolling friction, (12) is the same as the well-known condition for motion to be impossible or not unique (depending on the direction of the sliding velocity) /4/. If we use the coefficient $k$ of rolling friction, then (12) becomes ($\omega$ is the algebraic angular velocity of rolling)

$$f_* > |\zeta^{-1}(\rho^2\eta^{-1} + \eta + k\omega|\omega|^{-1})| \quad (\omega = \omega_1) \tag{13}$$

From (13) we can draw obvious conclusions about the influence of rolling friction (depending on the direction of rotation and on the size of $k$) on the conditions for TI to occur with plane motion.

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# THE SYNCHRONIZATION OF OSCILLATORS WHICH INTERACT VIA A MEDIUM*

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A system of $N$ non-linear oscilators which influence one another only as a result of their action on a common medium, is considered. The stability of the synchronized or partially synchronized periodic oscillations in the system is discussed. Special attention is paid to the case when $N \gg 1$. The problem of the synchronization of such oscillators, which do not interact directly but only indirectly via a common medium, is not new /1, 2/. It is usually assumed that the interaction is weak, so that the oscillators only slightly change their frequency and shape. The term "synchronization" usually means one of two effects: 1) the establishment of identical oscillations (in shape and phase) in a system of identical oscillators, 2) the establishment of a common period of oscillation in a system of identical or structurally similar oscillators. Biological problems which lead to a synchronization problem are considered in /3, 4/.

**1. The system and special features of the problem.** We consider $N$ objects of the same nature which have a stable selfexcited oscillatory mode in a band of fixed external conditions. The objects (oscillators) are located in a medium which they influence and thereby influence one another. We assume that the action of the oscillators on the medium is additive.

In biological applications, the number $N$ of objects is usually very large, and the influence of one oscillator on the medium is very small. For instance, if we are speaking of the biological oscillations inherent in a living cell, the action of an individual cell on the medium is proportional to the ratio of the cell volume $v$ to the volume $V$ of the medium in which there are no cells.

In the elementary case of identical oscillators, the equations describing the system can be written as (the dot denotes differentiation with respect to time $t$)

$$s^{\cdot} = \frac{1}{N}\sum_{k=1}^{N} g(s, x_k); \quad x_k^{\cdot} = h(s, x_k); \quad k = 1, \ldots, N \tag{1.1}$$

$$g(s, x) = g^{(0)}(s) + \gamma g^{(1)}(s, x), \ \gamma = Nv/V \tag{1.2}$$

Here, the vector $s$ refers to the medium and $x$ to the oscillators (in general, $\dim s \neq \dim x$), $g^{(0)}(s)$ describes the change in the medium regardless of its "filling", and $g^{(1)}(s, x)$ is the influence of the oscillators on the medium. It is assumed below that (given a fixed "density" $\gamma$) $g(s, x)$ is independent of $N$.

In biological (as distinct from technical) applications, synchronization is usually of interest when it sets in rapidly (in a fairly small number of periods) and is preserved when

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